

Total proper connection of graphs*

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Abstract

A graph is said to be *total-colored* if all the edges and the vertices of the graph is colored. A path in a total-colored graph is a *total proper path* if (i) any two adjacent edges on the path differ in color, (ii) any two internal adjacent vertices on the path differ in color, and (iii) any internal vertex of the path differs in color from its incident edges on the path. A total-colored graph is called *total-proper connected* if any two vertices of the graph are connected by a total proper path of the graph. For a connected graph G , the *total proper connection number* of G , denoted by $tpc(G)$, is defined as the smallest number of colors required to make G total-proper connected. These concepts are inspired by the concepts of proper connection number $pc(G)$, proper vertex connection number $pvc(G)$ and total rainbow connection number $trc(G)$ of a connected graph G . In this paper, we first determine the value of the total proper connection number $tpc(G)$ for some special graphs G . Secondly, we obtain that $tpc(G) \leq 4$ for any 2-connected graph G and give examples to show that the upper bound 4 is sharp. For general graphs, we also obtain an upper bound for $tpc(G)$. Furthermore, we prove that $tpc(G) \leq \frac{3n}{\delta+1} + 1$ for a connected graph G with order n and minimum degree δ . Finally, we compare $tpc(G)$ with $pvc(G)$ and $pc(G)$, respectively, and obtain that $tpc(G) > pvc(G)$ for any nontrivial connected graph G , and that $tpc(G)$ and $pc(G)$ can differ by t for $0 \leq t \leq 2$.

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1 Introduction

In this paper, all graphs considered are simple, finite and undirected. We refer to the book [2] for undefined notation and terminology in graph theory. A path in an edge-colored graph is a *proper path* if any two adjacent edges differ in color. An edge-colored graph is *proper connected* if any two vertices of the graph are connected by a proper path of the graph. For a connected graph G , the *proper connection number* of G , denoted by $pc(G)$, is defined as the smallest number of colors required to make G proper connected. Note that $pc(G) = 1$ if and only if G is a complete graph. The concept of $pc(G)$ was first introduced by Borozan et al. [3] and has been well-studied recently. We refer the reader to [1, 5, 8, 11] for more details.

As a natural counterpart of the concept of proper connection, the concept of proper vertex connection was introduced by the authors [7]. A path in a vertex-colored graph is a *vertex-proper path* if any two internal adjacent vertices on the path differ in color. A vertex-colored graph is *proper vertex connected* if any two vertices of the graph are connected by a vertex-proper path of the graph. For a connected graph G , the *proper vertex connection number* of G , denoted by $pvc(G)$, is defined as the smallest number of colors required to make G proper vertex connected. Especially, set $pvc(G) = 0$ for a complete graph G . Moreover, we have $pvc(G) \geq 1$ if G is a noncomplete graph.

Actually, the concepts of the proper connection and proper vertex connection were motivated from the concepts of the rainbow connection and rainbow vertex connection. For details about them we refer to a book [10] and a survey paper [9]. Here we only state the concept of the total rainbow connection of graphs, which was introduced by Liu et al. [12] and also studied in [6, 13]. A graph is *total-colored* if all the edges and vertices of the graph are colored. A path in a total-colored graph is a *total rainbow path* if all the edges and internal vertices on the path differ in color. A total-colored graph is *total rainbow connected* if any two vertices of the graph are connected by a total rainbow path of the graph. For a connected graph G , the *total rainbow connection number* of G , denoted by $trc(G)$, is defined as the smallest number of colors required to make G total rainbow connected. Motivated by the concept of the total rainbow connection, now for the proper connection and proper vertex connection we introduce the concept of the total proper connection. A path in a total-colored graph is a *total proper path* if (i) any two adjacent edges on the path differ in color, (ii) any two internal adjacent vertices on the path differ in color, and (iii) any internal vertex of the path differs in color from its incident edges on the path. A total-colored graph is *total proper connected* if any two vertices of the graph are connected by a total proper path of the graph. For a connected graph G , the *total proper connection number* of G , denoted by $tpc(G)$, is defined as the smallest number of

colors required to make G total proper connected. It is easy to obtain that $tpc(G) = 1$ if and only if G is a complete graph, and $tpc(G) \geq 3$ if G is not complete. Moreover,

$$tpc(G) \geq \max\{pc(G), pvc(G)\}. \quad (*)$$

We can also extend the definition of the total proper connection to that of the total proper k -connection $tpc_k(G)$ in a similar way as the definitions of the proper k -connection $pc_k(G)$, proper vertex k -connection $pvc_k(G)$ and total rainbow k -connection $trc_k(G)$, which were introduced by Borozan et al. in [3], the present authors in [7] and Liu et al. in [12], respectively. However, one can see that when k is larger very little have been known. Almost all known results are on the case for $k = 1$. So, in this paper we only focus our attention on the total proper connection $tpc(G)$ of graphs, i.e., $tpc_k(G)$ for the case $k = 1$.

The rest of this paper is organized as follows: In Section 2, we mainly determine the value of $tpc(G)$ for some special graphs, and moreover, we present some preliminary results. In Section 3, we obtain that $tpc(G) \leq 4$ for any 2-connected graph G and give examples to show that the upper bound 4 is sharp. For general graphs, we also obtain an upper bound for $tpc(G)$. In Section 4, we prove that $tpc(G) \leq \frac{3n}{\delta+1} + 1$ for a connected graph G with order n and minimum degree δ . In Section 5, we compare $tpc(G)$ with $pvc(G)$ and $pc(G)$, respectively, and obtain that $tpc(G) > pvc(G)$ for any nontrivial connected graph G , and that $tpc(G)$ and $pc(G)$ can differ by t for $0 \leq t \leq 2$.

2 Preliminary results

In this section, we present some preliminary results on the total proper connection number and determine the value of $tpc(G)$ when G is a nontrivial tree, a complete bipartite graph and a complete multipartite graph.

Proposition 1. *If G is a nontrivial connected graph and H is a connected spanning subgraph of G , then $tpc(G) \leq tpc(H)$. In particular, $tpc(G) \leq tpc(T)$ for every spanning tree T of G .*

Proposition 2. *Let G be a connected graph of order $n \geq 3$ that contains a bridge. If b is the maximum number of bridges incident with a single vertex in G , then $tpc(G) \geq b + 1$.*

Let $\Delta(G)$ denote the maximum degree of a connected graph G . We have the following.

Theorem 1. *If T is a tree of order $n \geq 3$, then $tpc(T) = \Delta(T) + 1$.*

Proof. Since each edge in T is a bridge, we have $tpc(T) \geq \Delta(T) + 1$ by Proposition 2. Now we just need to show that $tpc(T) \leq \Delta(T) + 1$. Let v be the vertex with maximum degree

$\Delta(T)$ and $N(v) = \{v_1, v_2, \dots, v_{\Delta(T)}\}$ denote its neighborhood. Take the vertex v as the root of T . Define a total-coloring c of T with $\Delta(T) + 1$ colors in the following way: Let u be a vertex in T . If $u = v$, color (i) v and its incident edges with distinct colors from $A = \{1, 2, \dots, \Delta(T), \Delta(T) + 1\}$, and (ii) v_i with the color from $A \setminus \{c(v), c(vv_i)\}$ for $1 \leq i \leq \Delta(T)$. If $u \neq v$, there exists a father of u , say u' . Let $N(u) = \{u', u_1, u_2, \dots, u_{d(u)-1}\}$ denote the neighborhood of u . Color the edges $\{uu_j : 1 \leq j \leq d(u) - 1\}$ with distinct colors from $A \setminus \{c(u), c(uu')\}$, and the vertex u_j with the color from $A \setminus \{c(u), c(uu_j)\}$ for $1 \leq j \leq d(u) - 1$.

For any two vertices x_1 and x_2 in T , let P_i be a path from x_i to v , where $i \in \{1, 2\}$. Next we shall show that there is a total proper path P between x_1 and x_2 . If P_1 and P_2 are edge-disjoint, then $P = x_1P_1vP_2x_2$; otherwise, we walk from x_1 along P_1 to the earliest common vertex, say y , and then switch to P_2 and walk to x_2 , i.e., $P = x_1P_1yP_2x_2$. Thus, $tpc(T) \leq \Delta(T) + 1$, and therefore, $tpc(T) = \Delta(T) + 1$. \square

The consequence below is immediate from Proposition 1 and Theorem 1.

Corollary 1. *For a nontrivial connected graph G ,*

$$tpc(G) \leq \min\{\Delta(T) + 1 : T \text{ is a spanning tree of } G\}.$$

A *Hamiltonian path* in a graph G is a path containing every vertex of G and a graph having a Hamiltonian path is a *traceable graph*. We get the following result.

Corollary 2. *If G is a traceable graph that is not complete, then $tpc(G) = 3$.*

Let $K_{m,n}$ denote a complete bipartite graph, where $1 \leq m \leq n$. Clearly, $tpc(K_{1,1}) = 1$ and $tpc(K_{1,n}) = n + 1$ if $n \geq 2$. For $m \geq 2$, we have the result below.

Theorem 2. *For $2 \leq m \leq n$, we have $tpc(K_{m,n}) = 3$.*

Proof. Let the bipartition of $K_{m,n}$ be U and V , where $U = \{u_1, \dots, u_m\}$ and $V = \{v_1, \dots, v_n\}$. Since $K_{m,n}$ is not complete, it suffices to show that $tpc(K_{m,n}) \leq 3$. Now we divide our discussion into two cases.

Case 1. $m = 2$.

We first give a total-coloring of $K_{m,n}$ with 3 colors. Color (1) the vertex u_1 and the edge v_1u_2 with color 1, (ii) the vertex u_2 and the edge u_1v_1 with color 2, and (iii) all the other edges and vertices with color 3. Then we show that there is a total proper path P between any two vertices u, v of $K_{m,n}$. It is clear that u and v are total proper connected by an edge if they belong to different parts of the bipartition. Next we consider that u and v are in the same part of the bipartition. For $u, v \in U$, we have $P = uv_1v$. For $u, v \in V$, if one of them is v_1 , then $P = uu_1v$; otherwise, $P = uu_1v_1u_2v$.

Case 2. $m \geq 3$.

Similarly, we first give a total-coloring of $K_{m,n}$ with 3 colors. Color the vertices and edges of the cycle $u_1v_1u_2v_2u_3v_3u_1$ starting from u_1 in turn with the colors 1, 2, 3. For $4 \leq i \leq n$ and $4 \leq j \leq m$, color (i) u_3v_i with color 1, (ii) u_jv_1 with color 2, and (iii) all the other edges and vertices with color 3. Now we show that there is a total proper path P between any two vertices u, v of $K_{m,n}$. It is clear that u and v are total proper connected by an edge if they belong to different parts of the bipartition. For $u, v \in U \setminus \{u_2, u_3\}$, we have $P = uv_1u_2v_2u_3v_3v$. For $u, v \in V \setminus \{v_1, v_2\}$, we have $P = uu_1v_1u_2v_2u_3v$. It can be checked that u and v are total proper connected in all other cases.

Therefore, the proof is complete. \square

Since any complete multipartite graph has a spanning complete bipartite subgraph, we obtain the following corollary.

Corollary 3. *If G is a complete multipartite graph that is neither a complete graph nor a tree, then $tpc(G) = 3$.*

3 Connectivity

In this section, we first prove that $tpc(G) \leq 4$ for any 2-connected graph G . Also we show that this upper bound is sharp by presenting a family of a 2-connected graphs. Finally, we state an upper bound of $tpc(G)$ for general graphs.

Given a colored path $P = v_1v_2 \dots v_{s-1}v_s$ between any two vertices v_1 and v_s , we denote by $start_e(P)$ the color of the first edge in the path, i.e., $c(v_1v_2)$, and by $end_e(P)$ the last color, i.e., $c(v_{s-1}v_s)$. Moreover, let $start_v(P)$ be the color of the first internal vertex in the path, i.e., $c(v_2)$, and $end_v(P)$ be the last color, i.e., $c(v_{s-1})$. If P is just the edge v_1v_s , then $start_e(P) = end_e(P) = c(v_1v_s)$, $start_v(P) = c(v_s)$, and $end_v(P) = c(v_1)$.

Definition 1. Let c be a total-coloring of G that makes G total proper connected. We say that G has the strong property if for any pair of vertices $u, v \in V(G)$, there exist two total proper paths P_1, P_2 between them (not necessarily disjoint) such that (1) $c(u) \neq start_v(P_i)$ and $c(v) \neq end_v(P_i)$ for $i = 1, 2$, and (2) both $\{c(u), start_e(P_1), start_e(P_2)\}$ and $\{c(v), end_e(P_1), end_e(P_2)\}$ are 3-sets.

Let G be a connected graph and H be a spanning subgraph of G . We say that H is a *spanning minimally 2-connected subgraph* of G if the removal of any edge from H would leave H 1-connected.

Theorem 3. *Let G be a 2-connected graph. Then $tpc(G) \leq 4$ and there exists a total-coloring of G with 4 colors such that G has the strong property.*

Proof. Let G' be a spanning minimally 2-connected subgraph of G . We apply induction on the number of ears in an ear-decomposition of G' . The base case is that G' is simply a cycle $C_n = v_1v_2 \dots v_nv_{n+1}(=v_1)$. Obviously, $tpc(C_3) = 1$ and $tpc(C_n) = 3$ for $n \geq 4$. Next define a total-coloring c of C_n with 4 colors by

$$c(v_iv_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd, } 1 \leq i \leq 2k-1 \text{ for } n = 2k \text{ or } n = 2k+1 \\ 2, & \text{if } i \text{ is even, } 2 \leq i \leq n \text{ for } n = 2k \text{ or } n = 2k+1 \\ 4, & \text{if } i = 2k+1 \text{ for } n = 2k+1 \end{cases} \quad (1)$$

and

$$c(v_i) = \begin{cases} 3, & \text{if } i \text{ is odd, } 1 \leq i \leq 2k-1 \text{ for } n = 2k \text{ or } n = 2k+1 \\ 4, & \text{if } i \text{ is even, } 2 \leq i \leq 2k \text{ for } n = 2k \text{ or } n = 2k+1 \\ 1, & \text{if } i = 2k+1 \text{ for } n = 2k+1. \end{cases} \quad (2)$$

Clearly, the total-coloring c makes G' have the strong property.

In an ear-decomposition of G' , let P be the last ear with at least one internal vertex since G' is assumed to be minimally 2-connected. And denote by G_1 the graph after removal of the internal vertices of P . Let u and v be the vertices of $P \cap G_1$ and then $P = uu_1u_2 \dots u_pv$. By induction hypothesis, there exists a total-coloring of G_1 with 4 colors such that G_1 is total proper connected with the strong property. We give such a total-coloring to G_1 . Then there exist two total proper paths P_1 and P_2 from u to v such that (1) $c(u) \neq start_v(P_i)$ and $c(v) \neq end_v(P_i)$ for $i = 1, 2$, and (2) both $\{c(u), start_e(P_1), start_e(P_2)\}$ and $\{c(v), end_e(P_1), end_e(P_2)\}$ are 3-sets. Let $A = \{1, 2, 3, 4\}$. Color the edge uu_1 with the color from $A \setminus \{c(u), start_e(P_1), start_e(P_2)\}$, and then total-properly color P from u to v so that $c(u_1) \neq c(u)$, $c(u_p) \neq c(v)$ and $c(u_pv) \neq c(v)$. If $c(u_pv) \notin \{end_e(P_1), end_e(P_2)\}$, it will become clear that this is the easier case, and so we consider the case that $c(u_pv) \in \{end_e(P_1), end_e(P_2)\}$ in the following.

Without loss of generality, suppose that $c(u_pv) = end_e(P_2)$. We will show that G' is total proper connected with the strong property under this coloring. For any two vertices of G_1 , there exist two total proper paths connecting them with the strong property by induction hypothesis. Since $P \cup P_1$ forms a total proper connected cycle, any two vertices in this cycle also have the desired paths. Assume that $x \in P \setminus \{u, v\}$ and $y \in G_1 \setminus P_1$. Next we will show that there are two total proper paths from x to y with the strong property.

Since $y, u \in G_1$, there exist two total proper paths P_{u_1} and P_{u_2} starting at y and ending at u with the strong property. Analogously, there exist two total proper paths P_{v_1} and P_{v_2} starting at y and ending at v with the strong property. Since these paths have the strong property, suppose that $Q_1 = xPu_{u_1}y$ and $Q_2 = xPvP_{v_1}y$ are total proper paths.

If $\text{end}_e(Q_1) \neq \text{end}_e(Q_2)$, then Q_1 and Q_2 are the desired pair of paths. Thus, assume that $\text{start}_e(P_{v_1}) = \text{start}_e(P_{u_1})$.

Then there exists a total proper walk $R_1 = xPuP_i v P_{v_2} y$ for some $i \in \{1, 2\}$ (suppose $i = 1$). If R_1 is a path, then R_1 and $R_2 = Q_2$ are the desired two paths. Otherwise, let z denote the vertex closest to y on P_{v_2} which is in $P_1 \cap P_{v_2}$. Now consider the path $R'_1 = xPuP_1 z P_{v_2} y$. If R'_1 is a total proper path, then R'_1 and R_2 are the desired two paths, and so we suppose that $\text{end}_e(uP_1 z) = \text{start}_e(zP_{v_2} y)$. Since P_1 and P_{v_2} are total proper paths, $c(z) \neq \text{start}_v(zP_{v_2} y)$, $c(z) \neq \text{start}_v(zP_1 v)$ and $\text{end}_e(vP_1 z) \neq \text{end}_e(uP_1 z)$. Then $\text{end}_e(vP_1 z) \neq \text{start}_e(zP_{v_2} y)$. Let $S_1 = xPvP_1 z P_{v_2} y$ and $S_2 = Q_1$. Obviously, S_1 and S_2 are two total proper paths. Note that $\text{end}_e(zP_{v_2} y) = \text{start}_e(P_{v_2}) \neq \text{start}_e(P_{v_1}) = \text{start}_e(P_{u_1})$. Thus, S_1 and S_2 have the strong property. Since $\text{tpc}(G) \leq \text{tpc}(G')$ by Proposition 1, we have $\text{tpc}(G) \leq 4$ and there exists a total-coloring of G with 4 colors such that G has the strong property. This completes the proof of Theorem 3. \square

In order to show that the bound obtained in Theorem 3 is sharp, we give a family of 2-connected graphs G with $\text{tpc}(G) = 4$ (see Figure 1).

Proposition 3. *Let G be the graph obtained from an even cycle by adding two ears which are as long as their interrupting segments respectively, such that each segment has 2^k ($k \geq 2$) edges. Then $\text{tpc}(G) = 4$.*

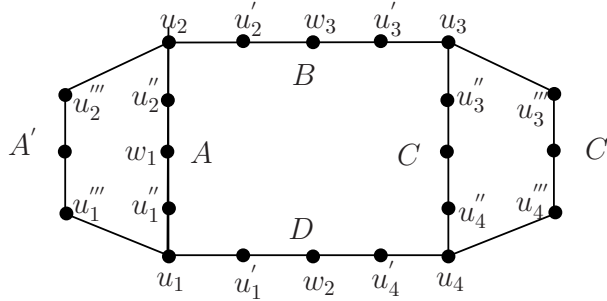


Figure 1: A 2-connected graph with $\text{tpc}(G) = 4$.

Before proving Proposition 3, we give the following fact.

Fact 1. *Let $C_n = v_1 v_2 \dots v_n v_{n+1} (= v_1)$. If there exists a total-coloring of C_n with three colors such that there are two total proper paths $v_j v_{j+1} \dots v_{i-1} v_i$ and $v_l v_{l+1} \dots v_{k-1} v_k$ where $1 \leq i < j < k < l \leq n$, $|i - l| > 1$ and $|k - j| > 1$, then $3 \mid n$.*

Proof of Proposition 3: Since $\text{tpc}(G) \leq 4$ by Theorem 3, we just need to prove that $\text{tpc}(G) \neq 3$. Assume that there is a total-coloring of G with 3 colors such that G is total

proper connected. Label the segments and some vertices of G as in Figure 1, where u'_i, u''_i and u'''_i are the neighbours of the vertex u_i for $i \in \{1, 2, 3, 4\}$.

Firstly, we shall show that the segments B and D are two total proper paths. If one of them is not, say B , then there is no total proper path in B from u_2 to u'_3 or from u_3 to u'_2 (say from u_2 to u'_3). Hence there exists a total proper path through D connecting u_2 and u'_3 , suppose $u_2ADC Bu'_3$ (this assumption, as opposed to using any of A' or C' , does not lose any generality). Next we consider the total proper path between u_1 and u'''_4 . Then there must exist a total proper path using the segments DC' or DCC' . If there is a total proper path $u_1DC' u'''_4$, then $c(u_4 u_4'') = c(u_4 u_4''')$. Thus the total proper path between u_4'' and u'''_4 is unique, i.e., $u_4''CC' u'''_4$, and then $c(u_3 u_3') = c(u_3 u_3''')$. However, we can not find a total proper path from u'_3 to u'''_3 , a contradiction. If there is a total proper path $u_1DCC' u'''_4$, then $c(u_3 u_3') = c(u_3 u_3''')$. Thus the total proper path connecting u'_3 and u'''_3 is unique, i.e., $u'_3BCC' u'''_3$. Then $u_3CC' u'''_3$ and $u_4CC' u'''_4$ are two total proper paths in $C \cup C'$ which is an even cycle of length 2^{k+1} , which contradicts Fact 1. Hence there is no total proper path from u_1 to u'''_4 , a contradiction. Therefore, the segments B and D are two total proper paths.

Secondly, we will show that at least one of A or A' must be total proper (and similarly, at least one of C or C'). Suppose both A and A' are not total proper. Then u_1 and u_2 are total proper connected by a path through C or C' , say $u_1DC Bu_2$. However, we can not find a total proper path connecting u_1 and u'''_4 in a similar discussion above, which is impossible. Thus, suppose A and C are total proper without loss of generality.

Finally, we know that at least one of the paths $u_1ABC u_4$ and $u_2ADC u_3$ must be not total proper by Fact 1. As we have shown, the only place which we can not go through is at the intersections, and so assume that the path $w_1AD w_2$ is not total proper, where $w_1 \in A \setminus \{u_1, u'_1, u_2, u'_2\}$ and $w_2 \in D \setminus \{u_1, u'_1, u_4, u'_4\}$. In the following, we consider the total proper path P from w_2 to w_1 and divide our discussion into two cases:

Case 1. P is w_2DCBAw_1 or $w_2DCBA'Aw_1$.

Between w_2 and u'''_4 , there must exist a total proper path P_1 . If P_1 is $w_2DC' u'''_4$, then $c(u_4 u_4'') = c(u_4 u_4''')$. Hence, there is only one total proper path $u_4''CC' u'''_4$ from u_4'' to u'''_4 ; otherwise $u_1DC Bu_2$ and $u_3BA' Du_4$ are two total proper paths in the cycle $A' \cup B \cup C \cup D$ for a contradiction. Then it follows that $c(u_3 u'_3) = c(u_3 u'''_3)$. Similarly, we can deduce that there is no total proper path connecting u'_3 and u'''_3 , which is impossible. If P_1 is $w_2DCC' u'''_4$, then $c(u_3 u'_3) = c(u_3 u'''_3)$. In a similar discussion, we obtain that the total proper path from u'_3 to u'''_3 is unique, i.e., $u'_3BCC' u'''_3$. Then $u_3CC' u'''_3$ and $u_4CC' u'''_4$ are two total proper paths in $C \cup C'$, a contradiction. If P_1 is $w_2DA'BC' u'''_4$ or $w_2DA'BCC' u'''_4$, then $u_1DC Bu_2$ and $u_3BA' Du_4$ are two total proper paths in $A' \cup B \cup C \cup D$, which again contradicts Fact 1.

Case 2. P is $w_2DA'Aw_1$.

Consider the total proper path P_2 from w_2 to w_3 , where $w_3 \in B \setminus \{u_2, u'_2, u_3, u'_3\}$. If P_2 is w_2DCBw_3 , then we can prove that this subcase could not happen in a similar way as Case 1. If P_2 is $w_2DA'Bw_3$, then $c(u_2u'_2) = c(u_2u''_2)$. From u'_2 to u''_2 , there is only one total proper path $u'_2BA'Au''_2$ since we can not go through w_2DCBw_3 . However, $u_2AA'u_2'''$ and $u_1A'Aw_1$ are two total proper paths in $A \cup A'$ for a contradiction.

The proof is thus complete. \square

Remark 1. Remember that for a 2-connected graph G , we have that the proper connection number $pc(G) \leq 3$; see [3]. But, if we consider a 2-connected bipartite graph G , then we have that $pc(G) = 2$. That means that the bipartite property can lower down the number of color by 1. However, from Proposition 3 we see that the bipartite property cannot play a role in general to lower down the number of colors for the total proper connection number, since the graphs in Proposition 3 are bipartite but their total proper connection numbers reach the upper bound 4.

Finally, we prove an upper bound of $tpc(G)$ for general graphs.

Theorem 4. *Let G be a connected graph and $\tilde{\Delta}$ denote the maximum degree of a vertex which is an endpoint of a bridge in G . Then $tpc(G) \leq \tilde{\Delta}(G) + 1$ if $\tilde{\Delta}(G) \geq 4$ and $tpc(G) \leq 4$ otherwise.*

In order to prove Theorem 4, we need a lemma below. Let $R(v)$ denote the set of colors presented on the vertex v and edges incident to v .

Lemma 1. *Let H be a graph obtained from a block B_0 with $V(B_0) = \{v_1, \dots, v_n\}$ by adding $t_i (\geq 0)$ nontrivial blocks and $s_i (\geq 0)$ pendant edges at v_i for $1 \leq i \leq n$. Consider $\tilde{\Delta}$ as the maximum degree of a vertex which is an endpoint of a bridge in H . Then $tpc(H) \leq \max\{\tilde{\Delta}(H) + 1, 4\}$.*

Proof. Let $k = \max\{\tilde{\Delta}(H) + 1, 4\}$ and $A = \{1, 2, \dots, k\}$. We give a total-coloring c of H using A as follows.

Step 1. If B_0 is a trivial block, then we give a total-coloring with 3 colors to B_0 such that $c(v_1), c(v_2), c(v_1v_2)$ are different from each other; otherwise, we give a total-coloring with 4 colors to B_0 that makes it have the strong property by Theorem 3. Let $L(v_i) = \{c(v_i), c(v_i) + 1, c(v_i) + 2, c(v_i) + 3\}$ modulo k for $1 \leq i \leq n$.

Step 2. For $1 \leq i \leq n$, if $t_i > 0$, then we give a total-coloring with 4 colors from $L(v_i)$ to each uncolored nontrivial block at v_i , denoted by B_j^i ($1 \leq j \leq t_i$), that makes each of them have the strong property by Theorem 3; afterwards if $s_i > 0$, then color s_i uncolored

pendant edges at v_i , denoted by $v_i v_1^i, \dots, v_i v_{s_i}^i$, with distinct colors from $A \setminus R(v_i)$ and then color each pendant vertex v_m^i using $A \setminus \{c(v_i), c(v_i v_m^i)\}$ for $1 \leq m \leq s_i$.

Next we show that H is total proper connected under the coloring c . If B_0 is a nontrivial block, then each pair of the vertices in B_0 has two total proper paths between them with the strong property. It will become clear that this is the easier case so we consider the case that B_0 is a trivial block. Let u and w be two vertices of H . It is obvious that there exists a total proper path connecting them if both belong to the same block. Suppose that $t_i > 0$ and $s_i > 0$ for $i = 1, 2$. If $u \in \cup_{j=1}^{t_1} B_j^1$ and $w \in \cup_{l=1}^{t_2} B_l^2$, then there exist two paths P_{u_1} and P_{u_2} from u to v_1 with the strong property. We know that $u P_{u_j} v_1 v_2$ is a total proper path for some $j \in \{1, 2\}$ (suppose $j = 1$). Similarly, there exists a total proper path $w P_{w_1} v_2 v_1$ from w to v_1 where P_{w_1} is a total proper path connecting w and v_2 . Thus, we can find a total proper path $u P_{u_1} v_1 v_2 P_{w_1} w$ between u and w . If $u \in \{v_m^1 : 1 \leq m \leq s_1\}$ and $w \in \{v_l^2 : 1 \leq l \leq s_2\}$, then $u v_1 v_2 w$ is a total proper path under the coloring c . For the other cases, it can be checked that there exists a total proper path connecting u and w in a similar way. Therefore, $tpc(H) \leq \max\{\tilde{\Delta}(H) + 1, 4\}$. \square

Now we are ready to prove Theorem 4.

Proof of Theorem 4: Let B_1, \dots, B_l be the blocks of G and $B(G)$ denote the block graph of G with vertex set $\{B_1, \dots, B_l\}$. Now, we consider a breadth-first search tree (BFS-tree) T of $B(G)$ with root B_1 and suppose that the blocks have an order B_1, \dots, B_l . Let $k = \max\{\tilde{\Delta}(G) + 1, 4\}$ and $A = \{1, 2, \dots, k\}$. We will give a total-coloring c using A in the following.

We give a total-coloring to B_1 and its neighbor blocks of G in a similar way as in Lemma 1. Then we can get that G is total proper connected if there are no more blocks in G . Hence, suppose that there are uncolored blocks in G . We extend our coloring from B_1 in a Breadth First Search way until there is no more blocks in G , i.e., if B_i has uncolored neighbor blocks, we give a total-coloring to its uncolored neighbor blocks of G in a similar way as **Step 2**; otherwise, consider B_{i+1} .

Now we prove that G is total proper connected. Let u and w be two vertices in G . It is obvious that there exists a total proper path between them if both belong to the same block. Suppose that $u \in B_i$ and $w \in B_j$ ($i \neq j$). Let P denote the path from B_i to B_j in the BFS-tree T . Then we can find a total proper path from u to w traversing the blocks on P under the coloring c . Therefore, $tpc(G) \leq \tilde{\Delta}(G) + 1$ if $\tilde{\Delta}(G) \geq 4$ and $tpc(G) \leq 4$ otherwise. \square

4 Minimum degree

In this section, we prove the following result concerning the minimum degree.

Theorem 5. *Let G be a connected graph of order n with minimum degree δ , then $\text{tpc}(G) \leq \frac{3n}{\delta+1} + 1$.*

Given a graph G , a set $D \subseteq V(G)$ is called a *two-step dominating set* of G if every vertex in G which is not dominated by D has a neighbor that is dominated by D . Moreover, a two-step dominating set D is called a *two-way two-step dominating set* if (a) every pendant vertex of G is included in D , and (b) every vertex in $N^2(D)$ has at least two neighbors in $N^1(D)$, where $N^k(D)$ denotes the set of all vertices at distance exactly k from D . Further, if $G[D]$ is connected, D is called a *connected two-way two-step dominating set* of G .

Lemma 2. [4] *Every connected graph G of order $n \geq 4$ and minimum degree δ has a connected two-way two-step dominating set D of size at most $\frac{3n}{\delta+1} - 2$.*

Proof of Theorem 5: The proof goes similarly as that of the main result in [11] by Li et al.

We are given a connected graph G of order n with minimum degree δ . The assertion can be easily verified for $n \leq 3$ and so suppose $n \geq 4$. Let D denote a connected two-way two-step dominating set of G and $k = |D|$. Then we have $k \leq \frac{3n}{\delta+1} - 2$ by Lemma 2. Let $F(x) = \{u : u \text{ is a neighbor of } x \text{ in } D\}$ for $x \in N^1(D)$ and $F'(y) = \{u : u \text{ is a neighbor of } y \text{ in } N^1(D)\}$ for $y \in N^2(D)$.

Case 1. For each vertex $y \in N^2(D)$, its neighbors in $N^1(D)$ has at least one common neighbor in D , i.e., $\cap_{x \in F'(y)} F(x) \neq \emptyset$.

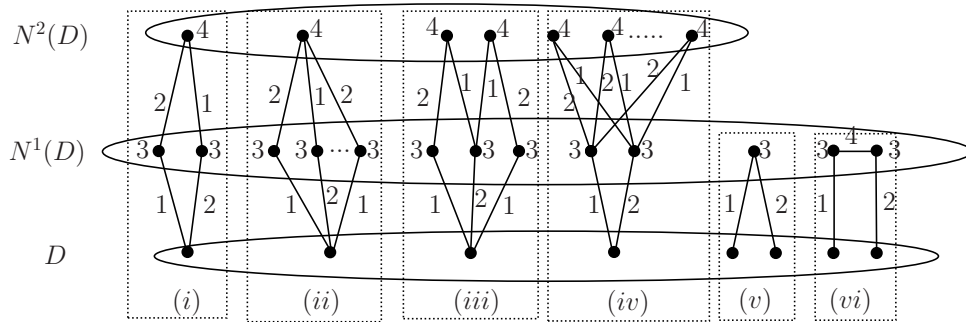


Figure 2: The total-coloring for the spanning subgraph G_0 of G .

We consider a spanning subgraph $G_0 = p(i) \cup q(ii) \cup r(iii) \cup s(iv) \cup t(v) \cup z(vi) \cup G[D]$ of G (see Figure 2, where $p(i)$ denotes the union of p graphs each of which is isomorphic

to the graph (i) and similarly for $q(ii), r(iii), s(iv), t(v)$ and $z(vi)$). Next, we give a total-coloring c to G_0 using $\{1, 2, \dots, k, k+1, k+2, k+3\}$. For the edges and vertices of $G[D]$, let T be a spanning tree of $G[D]$. Then by Theorem 1, T can be total-colored using $\{4, 5, \dots, k, k+1, k+2, k+3\}$ such that for each edge $uv \in E(T)$, the colors of u, v and uv are different from each other. We color T in such a way and the edges of $G[D] \setminus T$ with any used colors (denote this coloring of $G[D]$ by c_D). For the other edges and vertices in G_0 , color them as depicted in Figure 2.

Since each pair of vertices $u, w \in D$ has a total proper path P connecting them such that $c(u) \notin \{start_v(P), start_e(P)\}$ and $c(w) \notin \{end_v(P), end_e(P)\}$, it suffices to show that G_0 is total proper connected in the assumption that $\cap_{y \in N^2(D)} \{F(x) : x \in F'(y)\} = \{w\}$. Take any two vertices u and v in $V(G_0)$. If $u, v \in N^2(D)$, then u has a neighbor u' in $N^1(D)$ and similarly v has a neighbor v' in $N^1(D)$. Hence, if $c(u'w) \neq c(v'w)$, $uu'wv'v$ is a total proper path; otherwise, $uu''wv'v$ is a total proper path where u'' is another neighbor of u in $N^1(D)$. It is easy to check that u and v are total proper connected in all other cases.

Case 2. There exists one vertex $y \in N^2(D)$ whose neighbors in $N^1(D)$ has no common neighbors in D , i.e., $\cap_{x \in F'(y)} F(x) = \emptyset$.

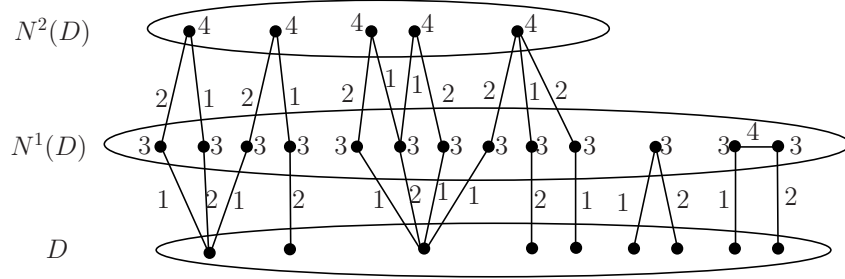


Figure 3: An example for the total-coloring for the spanning subgraph G_0 of G .

We consider a spanning subgraph G_0 of G (see Figure 3). Next, we give a total-coloring c to G_0 using $\{1, 2, \dots, k, k+1, k+2, k+3\}$. For the edges and vertices in $G[D]$, we use the total-coloring c_D as in Case 1. For any vertex $v \in N^2(D)$, color vx_1 with color 1 and x_1u_1 with color 2 where $x_1 \in F'(v)$ and $u_1 \in F(x_1)$. And then color vx_i with color 2 and x_iu_i with color 1 where $x_i \in F'(v) \setminus \{x_1\}$ and $u_i \in F(x_i)$. For any vertex $v \in N^1(D) \setminus \cup_{y \in N^2(D)} F'(y)$, color the edges incident to v as depicted in Figure 3. Moreover, we color the vertices of $N^1(D)$ with color 3 and the vertices of $N^2(D)$ with color 4.

Now we show that G_0 is total proper connected. Take any two vertices u and v in $V(G_0)$. If $u, v \in N^2(D)$, there exist two paths $uu'u''$ and $vv'v''$ connecting to D , where

$u', v' \in N^1(D)$ and $u'', v'' \in D$. Thus, if $u'' \neq v''$, u and v are total proper connected by a path $uu'u''Pv''v'v$ where P is a total proper path from u'' to v'' in $G[D]$; otherwise, there exists a total proper path connecting u and v in a similar discussion as Case 1. It can be checked that u and v are total proper connected in all other cases. Therefore, we have $tpc(G) \leq tpc(G_0) \leq \frac{3n}{\delta+1} - 2 + 3 = \frac{3n}{\delta+1} + 1$ by Proposition 1 and Lemma 2. \square

5 Compare $tpc(G)$ with $pvc(G)$ and $pc(G)$

Let G be a nontrivial connected graph. Recall that $tpc(G) \geq \max\{pc(G), pvc(G)\}$. The question we may ask is, how tight are the inequalities $tpc(G) \geq pc(G)$ and $tpc(G) \geq pvc(G)$? By [7, Proposition 1 and Theorem 1], we have that (1) $pvc(G) = 0$ if and only if G is a complete graph, (2) $pvc(G) = 1$ if and only if $diam(G) = 2$, and (3) $pvc(G) = 2$ if and only if $diam(G) \geq 3$. Note that $tpc(G) = 1$ if and only if G is a complete graph, and $tpc(G) \geq 3$ if G is not complete. Thus, it follows that $tpc(G) > pvc(G)$.

Next we consider the tightness of the inequality $tpc(G) \geq pc(G)$. Observe that $tpc(G) = pc(G) = 1$ if and only if G is a complete graph. Proposition 4 below shows that there exists an example graph G such that $tpc(G) = pc(G) = 3$.

Proposition 4. *Let G be the graph obtained from a cycle by adding three ears of length 3 such that each segment of the cycle has $6t$ ($t \geq 1$) edges. Then $tpc(G) = pc(G) = 3$.*

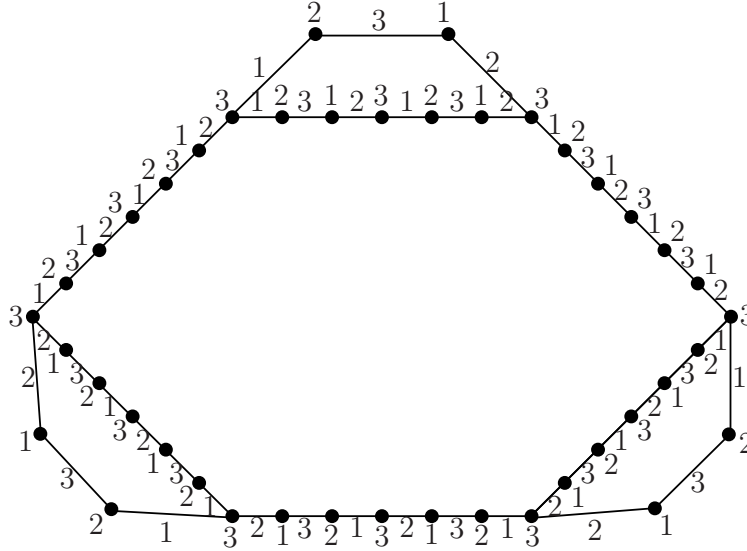


Figure 4: A 3-coloring of edges and vertices of G .

Proof. It can be verified that $pc(G) = 3$ by [3, Proposition 3]. Thus, it suffices to show

that $tpc(G) \leq 3$ by Ineq.(*). A 3-coloring of edges and vertices of G is shown in Figure 4 to make G total proper connected. Hence, we have that $tpc(G) = pc(G) = 3$. \square

However, we cannot show whether there exists a graph G such that $tpc(G) = pc(G) = k$ for any $k \geq 4$. Thus, we propose the following problem.

Problem 1. *For $k \geq 4$, does there exist a graph G such that $tpc(G) = pc(G) = k$?*

Now we consider the difference between $tpc(G)$ and $pc(G)$. If T is a tree of order $n \geq 3$, then $pc(T) = \Delta(T)$ by [1, Proposition 2.3] and $tpc(T) = \Delta(T) + 1$ by Theorem 1. Hence, $tpc(T) = pc(T) + 1$. Moreover, there exists an example graph depicted as in Proposition 3 such that $tpc(G) = pc(G) + 2$ since $tpc(G) = 4$ and $pc(G) = 2$ (we give a 2-edge-coloring of G by coloring alternately the edges of the segments A', C' and the cycle $A \cup B \cup C \cup D$). However, we have not found any graph G such that $tpc(G)$ and $pc(G)$ can differ by t ($t \geq 3$). Thus, we pose the following problem.

Problem 2. *For $t \geq 3$, does there exist a graph G such that $tpc(G) = pc(G) + t$?*

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